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Asymptotics for L_p Extremal Polynomials on the Unit Circle

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Let $p>1$, and $d\mu$ a positive finite Borel measure on the unit circle $\Gamma := \{z \in \mathbf{C} : |z| = 1\}$. Define the monic polynomial $\phi_{n,p}(z) = z^n + \cdots \in \mathscr{P}_n$ (the set of polynomials of degree at most n) satisfying

$$\int_\Gamma |\phi_{n,p}(z)|^p \, d\mu = \inf_{P \in \mathscr{P}_{n-1}} \int_\Gamma |z^n + P|^p \, d\mu.$$

Under certain conditions on $d\mu$, the asymptotics of $\phi_{n,p}(z)$ for z outside, on, or inside Γ are obtained (cf. Theorems 2.2 and 2.4). Zero distributions of $\phi_{n,p}$ are also discussed (cf. Theorems 3.1 and 3.2).

1. INTRODUCTION

Let $d\mu$ be a finite positive Borel measure on $\Gamma := \{z \in \mathbf{C} : |z| = 1\}$. Let $\mathscr{P}_n$ be the set of algebraic polynomials of degree at most n. For $p>0$, define $\phi_{n,p}(z) = z^n + \cdots \in \mathscr{P}_n$ satisfying

$$\|\phi_{n,p}\|_{L_p(d\mu)} = \inf_{P \in \mathscr{P}_{n-1}} \|z^n + P\|_{L_p(d\mu)} =: \varepsilon_{n,p},$$

where (and from now on) $\|g\|_{L_p(d\mu)} := ((1/2\pi) \int_\Gamma |g(z)|^p \, d\mu)^{1/p}$. We will consider the asymptotic behavior of $\phi_{n,p}(z)$ (outside or on Γ) and related problems. The motivation of this paper is a series of recent results obtained by Lubinsky and Saff concerning the asymptotics of monic polynomials $T_{n,p}(W, x)$ of minimal L_p norm associated with weight W on $[-1, 1]$ or

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R (cf. [7–12]). Under suitable conditions on $d\mu$, the nth root asymptotics of $\phi_{n,p}$ can be obtained as a special case from the results in the well-known paper by Fekete and Walsh [3].

As it is well-known in the theory of asymptotics of orthogonal polynomials (e.g., see [16]), we often first derive the asymptotic results for the orthogonal polynomials on the unit circle Γ and then transfer the results to the orthogonal polynomials on $[-1, 1]$. We wonder if this procedure can be adopted for the study of the asymptotic problems for $T_{n,p}(W, x)$ on $[-1, 1]$. In order to do so, we must solve the following two problems:

(i) establish the results for the unit circle case;

(ii) find the relation between $\phi_{n,p}$ and $T_{n,p}$ and transfer the results to $T_{n,p}$.

We only consider the problem (i) here. The second problem is still open. Set

$$\mu(\theta) := \int_{\{z : z = e^{it},\, 0 \leq t \leq \theta\}} d\mu,$$

then $\mu'(\theta)$ exists a.e. on $[0, 2\pi]$. Define the Szegő function of $d\mu$ by

$$D(d\mu, z) := \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(\theta)\, \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta\right\}, \qquad |z| < 1$$

(when $\log \mu'$ is not integrable, we define $D(d\mu, z) \equiv 0$). It can be seen that $D(d\mu, 0) \neq 0$ iff $\log \mu' \in L_1$.

When $\log \mu' \in L_1$, we say $d\mu$ satisfies *the Szegő condition*, and in this case we have the following:

(i) $D(d\mu, \cdot) \in H^2$ in the unit disk;

(ii) $D(d\mu, z) \neq 0$ for $|z| < 1$;

(iii) $D(d\mu, 0) > 0$;

(iv) $\lim_{r \to 1^-} D(d\mu, re^{i\theta}) =: D(d\mu, e^{i\theta})$ exists for almost every $\theta \in [0, 2\pi]$ and $|D(d\mu, e^{i\theta})|^2 = \mu'(\theta)$ a.e. on $[0, 2\pi]$ (cf. [16, p. 276]).

Define the geometric mean $G(d\mu)$ of $d\mu$ by (cf. [16, p. 275])

$$G(d\mu) := \{D(d\mu, 0)\}^2 = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log \mu'(\theta)\, d\theta\right\}$$

for $d\mu$ satisfying the Szegő condition; $G(d\mu) = 0$ otherwise.

2. Asymptotics for $\phi_{n,p}(z)$

We know the following result due to Szegő, Kolmogorov, and Krein (cf. [6, Chap. III] or [4, p. 270]).

THEOREM 2.1. *For every $p > 0$, we have*

$$\lim_{n \to \infty} \varepsilon_{n,p} = G(d\mu)^{1/p}. \tag{2.1}$$

It is easy to see that $\{\varepsilon_{n,p}\}_{n=0}^{\infty}$ is non-increasing, and so Theorem 2.1 tells us what the limit is. But it does not tell us the *rate* of the convergence. We will state and prove some results about the rate of convergence later (cf. Theorem 2.4). By modifying the methods of Szegő (cf. [16, Chap. XII]), we can obtain the following theorem.

THEOREM 2.2. *Suppose $d\mu$ satisfies the Szegő condition. Then for every $p > 1$ we have*

$$\phi_{n,p}(z) \cong G(d\mu)^{1/p} z^n [\bar{D}(d\mu, z^{-1})]^{-2/p} \qquad (n \to \infty) \tag{2.2}$$

locally uniformly for $|z| > 1$, where

$$\bar{D}(d\mu, z^{-1}) := \overline{D\left(d\mu, \frac{1}{z}\right)}.$$

Remark 1. Theorem 2.2 is a special case of Theorem 7.1 as stated in Geronimus' paper [5]. Geronimus considered (among other things) the asymptotics for the extremal polynomials of minimum L_p-norm taken over a rectifiable Jordan curve in the complex plane. We present the following more informative proof, which yields a useful inequality (see the remark after the proof of Theorem 2.2). We will need this inequality to characterize the measures satisfying *analytic condition* (cf. Theorem 3.1 and its proof).

Remark 2. For $p = 2$, Theorem 2.2 asserts the well-known asymptotic result for orthogonal polynomials (cf. [16, p. 297]).

Denote

$$\Phi_{n,p}(z) := \frac{\phi_{n,p}(z)}{\varepsilon_{n,p}},$$

then $\|\Phi_{n,p}\|_{L_p(d\mu)} = 1$. For $p(z) = a_n z^n + \cdots \in \mathscr{P}_n$, $a_n \neq 0$, let

$$p_n^*(z) = z^n \overline{p_n\left(\frac{1}{\bar{z}}\right)}.$$

Proof of Theorem 2.2. Following Szegő's idea (cf. [16, p. 302]), consider

$$[D(d\mu, z)]^{2p}\Phi^*_{n,p}(z)$$

which is analytic and has expansion

$$[G(d\mu)]^{1/p}/\varepsilon_{n,p} + d_{n,1}z + \cdots$$

for $|z| < 1$. First note that

$$\begin{aligned} I_n &:= \frac{1}{2\pi}\int_0^{2\pi} |[D(d\mu, e^{i\theta})]^{2/p}\Phi^*_{n,p}(e^{i\theta}) - 1|^2\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} |[D(d\mu, e^{i\theta})]^{2/p}\Phi^*_{n,p}(e^{i\theta})|^2\, d\theta \\ &\quad - 2Re\left\{\frac{1}{2\pi}\int_0^{2\pi} [D(d\mu, e^{i\theta})]^{2/p}\Phi^*_{n,p}(e^{i\theta})\, d\theta\right\} + 1 \\ &= \frac{1}{2\pi}\int_0^{2\pi} |[D(d\mu, e^{i\theta})]^{2/p}\Phi^*_{n,p}(e^{i\theta})|^2\, d\theta + 1 - 2\frac{[G(d\mu)]^{1/p}}{\varepsilon_{n,p}}, \qquad (2.3) \end{aligned}$$

where in the last equality we used the Cauchy formula. (Note that $D(d\mu, z)^{2/p} \in H^p$, so we can use the Cauchy formula for H^p (cf. [2, Sect. 3.3. Theorem 6]).) Next, for $p \geqslant 2$, by Hölder's inequality,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} |[D(d\mu, e^{i\theta})]^{2/p}\Phi^*_{n,p}(e^{i\theta})|^2\, d\theta &= \frac{1}{2\pi}\int_0^{2\pi} |[\mu'(\theta)]^{1/p}\Phi_{n,p}(e^{i\theta})|^2\, d\theta \\ &\leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} \mu'(\theta)\,|\Phi_{n,p}(e^{i\theta})|^p\, d\theta\right)^{2/p}\left(\frac{1}{2\pi}\int_0^{2\pi} d\theta\right)^{1-2/p} \\ &\leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} |\Phi_{n,p}(e^{i\theta})|^p\, d\mu(\theta)\right)^{2/p} = 1. \qquad (2.4) \end{aligned}$$

So by (2.1), (2.3), and (2.4)

$$I_n \leqslant 2 - 2\frac{[G(d\mu)]^{1/p}}{\varepsilon_{n,p}}.$$

Hence, by using Theorem 2.1 and the Cauchy formula, we have

$$\Phi^*_{n,p}(z) \cong [D(d\mu, z)]^{-2/p} \qquad (n \to \infty) \qquad (2.5)$$

locally uniformly for $|z| < 1$, or equivalently by using Theorem 2.1,

$$\phi_{n,p}(z) \cong [G(d\mu)]^{1/p} z^n \bar{D}(d\mu, z^{-1})^{-2/p} \qquad (n \to \infty)$$

locally uniformly for $|z| > 1$. So we have shown that (2.2) holds for $p \geqslant 2$.

We now consider the remaining case when $p \in (1, 2)$. Since

$$[D(d\mu, z)]^{2/p} \Phi^*_{n,p}(z) \in H^p,$$

the Hausdorff–Young inequality for the Taylor coefficients of a function in H^p space yields (cf. [2, Sect. 6.1])

$$\begin{aligned}(|[G(d\mu)]^{1/p}/\varepsilon_{n,p}|^q + |d_{n,1}|^q + \cdots)^{1/q} &\leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} |[D(d\mu, e^{i\theta})]^{2/p} \Phi^*_{n,p}(e^{i\theta})|^p \, d\theta\right)^{1/p} \\ &\leqslant 1,\end{aligned}$$

where q satisfies $1/p + 1/q = 1$, and so

$$|d_{n,1}|^q + |d_{n,2}|^q + \cdots \leqslant 1 - ([G(d\mu)]^{1/p}/\varepsilon_{n,p})^q.$$

Hence, for $|z| < 1$,

$$\begin{aligned}&|[D(d\mu, z)]^{2/p} \Phi^*_{n,p}(z) - 1| \\ &\quad\leqslant |[G(d\mu)]^{1/p}/\varepsilon_{n,p} - 1| + |d_{n,1} z + d_{n,2} z^2 + \cdots| \\ &\quad\leqslant |[G(d\mu)]^{1/p}|\varepsilon_{n,p} - 1| + (|d_{n,1}|^q + |d_{n,2}|^q + \cdots)^{1/q} \left(\frac{|z|^p}{1 - |z|^p}\right)^{1/p} \\ &\quad\leqslant |[G(d\mu)]^{1/p}/\varepsilon_{n,p} - 1| + |1 - ([G(d\mu)]^{1/p}/\varepsilon_{n,p})^q|^{1/q} \frac{|z|}{(1 - |z|^p)^{1/p}}.\end{aligned}$$

So again we get (2.5). This completes the proof of Theorem 2.2. ∎

Remark. In the above proof, we can see the following inequality holds when $p > 1$ and $d\mu$ satisfies the Szegő condition:

$$\max_{|z| \leqslant \rho} |\Phi^*_{n,p}(z) - [D(d\mu, z)]^{-2/p}| \leqslant K_\rho(\varepsilon_{n,p} - [G(d\mu)]^{1/p})^{1/p})^{1/\tilde{q}}.$$

Here $\rho \in (0, 1)$, $1/p + 1/q = 1$, $\tilde{q} = \max(2, q)$, and K_ρ is a constant only depending on ρ.

Now we turn our attention to the asymptotics of $\phi_{n,p}(z)$, or equivalently $\Phi_{n,p}(z)$, for z on and inside the unit circle. From now on, we will only consider the case when $d\mu$ is absolutely continuous, i.e., $d\mu(\theta) = \mu' \, d\theta$.

DEFINITION. (i) Let $s \geqslant 0$ be an integer and $\alpha \in (0, 1)$. We say that $d\mu$ *satisfies an* (s, α)*-Lipschitz condition* if $d\mu$ is absolutely continuous and

$$g(\theta) := [\bar{D}(d\mu, e^{-i\theta})]^{-2/p} \qquad (\theta \in [0, 2\pi])$$

has sth derivative and the sth derivative satisfies a Lipschitz condition of order α.

(ii) Let $r > 1$; we say that $d\mu$ *satisfies an analytic condition for* r if $D(d\mu, z)^{-2/p}$ has analytic continuation to $|z| < r$.

Let us first state the following Lemma 2.3 which is a special case of the known results for weighted Faber polynomials (cf. [15]). Define the polynomials F_n as the principal (polynomial) part of

$$z^n[\bar{D}(d\mu, z^{-1})]^{-2/p}$$

at ∞ for $n \geqslant 0$, or equivalently, define F_n as follows: for $|z| < R$,

$$F_n(z) := \frac{1}{2\pi i}\int_{|\zeta| = R} \frac{\zeta^n[\bar{D}(d\mu, \zeta^{-1})]^{-2/p}}{\zeta - z}\, d\zeta, \tag{2.6}$$

where $R > 1$.

LEMMA 2.3 [15, p. 9]. (i) *Let s be a non-negative integer and $\alpha \in (0, 1)$. If $d\mu$ satisfies an (s, α)-Lipschitz condition, then*

$$F_n(z) = z^n[\bar{D}(d\mu, z^{-1})]^{-2/p} + O\left(\frac{\ln n}{n^{\alpha + s}}\right)$$

uniformly for $|z| \geqslant 1$.

(ii) *Let $r > 1$; if $d\mu$ satisfies an analytic condition for r, then for every* $r_1 \in (1, r)$,

$$F_n(z) = z^n[\bar{D}(d\mu, z^{-1})]^{-2/p} + O\left(\frac{1}{r_1^n}\right)$$

uniformly for $|z| \geqslant r_1^{-1}$.

Now we can state

THEOREM 2.4. (i) *Let $s \geqslant 0$ be an integer, $\alpha \in (0, 1)$. If $d\mu$ satisfies an (s, α)-Lipschitz condition, then*

$$\varepsilon_{n,p} = [G(d\mu)]^{1/p} + O\left(\frac{\ln n}{n^{s+\alpha}}\right), \tag{2.7}$$

and if, in addition,

$$s\begin{cases}\geqslant 1 & \text{for} \quad p\geqslant 2\\ >(q-1-\alpha) & \text{for} \quad 1<p<2 \quad (1/p+1/q=1),\end{cases}$$

then we have

$$\Phi_{n,p}(z)=z^n[\bar{D}(d\mu,z^{-1})]^{-2/p}+\beta_n(z), \tag{2.8}$$

where

$$|\beta_n(z)|\leqslant\begin{cases}C\left(\dfrac{\ln n}{n^{\alpha+s-1}}\right)^{1/2} & \text{for} \quad p\geqslant 2,\\ C\left(\dfrac{\ln n}{n^{\alpha+s+1-q}}\right)^{1/q} & \text{for} \quad 1<p<2 \quad (1/p+1/q=1),\end{cases}$$

uniformly for $|z|=1$.

(ii) *Let* $r>1$; *if* $d\mu$ *satisfies an analytic condition for* r, *then, for every* $r_1\in(1,r)$,

$$\varepsilon_{n,p}=[G(d\mu)]^{1/p}+O\left(\frac{1}{r_1^n}\right), \tag{2.9}$$

and for some $r_2\in(1,r)$,

$$\Phi_{n,p}(z)=z^n[\bar{D}(d\mu,z^{-1})]^{-2/p}+O\left(\frac{1}{r_2^n}\right), \tag{2.10}$$

locally uniformly for $|z|>r^{-1}$.

Proof. We first show (2.7) and (2.9). Let

$$e_{n,p}:=\|[G(d\mu)]^{1/p}\{F_n(z)-z^n[\bar{D}(d\mu,z^{-1})]^{-2/p}\}\|_{L_p(d\mu)}.$$

By the definition of $\varepsilon_{n,p}$,

$$\begin{aligned}\varepsilon_{n,p}&\leqslant\|[G(d\mu)]^{1/p}F_n\|_{L_p(d\mu)}\\ &\leqslant[G(d\mu)]^{1/p}\,\|z^n[\bar{D}(d\mu,z^{-1})]^{-2/p}\|_{L_p(d\mu)}+O(e_{n,p})\\ &=[G(d\mu)]^{1/p}+O(e_{n,p}).\end{aligned}$$

But on the other hand,

$$\begin{aligned}\varepsilon_{n,p}^p&=\frac{1}{2\pi}\int_0^{2\pi}|\phi_{n,p}(e^{i\theta})|^p\,d\mu(\theta)\geqslant\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\phi_{n,p}(e^{i\theta})}{[\bar{D}(d\mu,e^{-i\theta})]^{-2/p}\,e^{in\theta}}\right|^p d\theta\\ &\geqslant\lim_{z\to\infty}\left|\frac{\phi_{n,p}(z)}{z^n[\bar{D}(d\mu,z^{-1})]^{-2/p}}\right|^p=G(d\mu).\end{aligned}$$

Thus

$$\varepsilon_{n,p} = [G(d\mu)]^{1/p} + O(e_{n,p}). \tag{2.11}$$

Now, by using Lemma 2.3, we can easily get the estimates for $e_{n,p}$ and establish (2.7) and (2.9), respectively.

Next we show (2.8) and (2.10). Write

$$\Phi_{n,p}(z) = \lambda_0 F_0(z) + \cdots + \lambda_n F_n(z). \tag{2.12}$$

Then by comparing the coefficients of z^n on both sides in (2.12), we get

$$\frac{1}{\varepsilon_{n,p}} = \lambda_n [G(d\mu)]^{-1/p},$$

so by (2.11),

$$\begin{aligned}\lambda_n &= 1 - \frac{O(e_{n,p})}{[G(d\mu)]^{1/p} + O(e_{n,p})} \\ &= 1 + O(e_{n,p}).\end{aligned} \tag{2.13}$$

Note that by the definition of F_k and (2.12), we have

$$[\bar{D}(d\mu, z^{-1})]^{2/p} \Phi_{n,p}(z) = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \cdots + \gamma_1 z^{-1} + \cdots.$$

Now let us first assume $p \geqslant 2$, then

$$\begin{aligned}|\lambda_0|^2 + |\lambda_1|^2 + \cdots + |\lambda_n|^2 &\leqslant \frac{1}{2\pi} \int_0^{2\pi} |[\bar{D}(d\mu, e^{-i\theta})]^{2/p} \Phi_{n,p}(e^{i\theta})|^2 \, d\theta \\ &\leqslant \left(\frac{1}{2\pi} \int_0^{2\pi} |\Phi_{n,p}(e^{i\theta})|^p \, d\mu \right)^{2/p} = 1.\end{aligned}$$

Together with (2.13), this yields

$$|\lambda_0|^2 + \cdots + |\lambda_{n-1}|^2 = O(e_{n,p}). \tag{2.14}$$

For $p \in (1, 2)$, again we use the Hausdorff–Young inequality for

$$[D(d\mu, z)]^{2/p} \Phi^*_{n,p}(z) \in H^p.$$

Since

$$[D(d\mu, z)]^{2/p} \Phi^*_{n,p}(z) = \bar{\lambda}_n + \bar{\lambda}_{n-1} z + \cdots + \bar{\lambda}_0 z^n + \bar{\gamma}_1 z^{n+1} + \cdots,$$

so for q satisfying $1/p+1/q=1$, we have

$$(|\lambda_0|^q+|\lambda_1|^q+\cdots+|\lambda_n|^q)^{1/q}$$
$$\leqslant\left(\frac{1}{2\pi}\int_0^{2\pi}|[D(d\mu, e^{i\theta})]^{2/p}\Phi_{n,p}^*(e^{i\theta})|^p\,d\theta\right)^{1/p}\leqslant 1.$$

Hence, with (2.13), it follows that

$$|\lambda_0|^q+\cdots+|\lambda_{n-1}|^q=O(e_{n,p}). \tag{2.15}$$

Now by (2.14) and (2.15), it is easy to show that, for $p>1$,

$$|\lambda_0F_0(z)+\cdots+\lambda_{n-1}F_{n-1}(z)|=O(n^{1/\tilde{p}}e_{n,p}^{1/\tilde{q}}), \tag{2.16}$$

uniformly for $|z|=1$, where $\tilde{q}=\max(2, q)$, $1/\tilde{p}+1/\tilde{q}=1$ and $1/p+1/q=1$. Let $\tilde{e}_{n,p}=n^{1/\tilde{p}}e_{n,p}^{1/\tilde{q}}$, then by (2.12) and (2.16)

$$\Phi_{n,p}(z)-\lambda_nF_n(z)=O(\tilde{e}_{n,p}),$$

uniformly for $|z|=1$.

By Lemma 2.3, (2.13) and (2.16) we have

$$\Phi_{n,p}(z)=z^n[\bar{D}(d\mu, z^{-1})]^{-2/p}+O(\tilde{e}_{n,p}), \tag{2.17}$$

uniformly for $|z|=1$.

Finaly, if $d\mu$ satisfies the Lipschitz condition for (s, α), then it is easy to estimate $\tilde{e}_{n,p}$ and so to get (2.8) from (2.17). If $d\mu$ satisfies analytic condition, then by Lemma 2.3(ii), (2.17) holds uniformly for $|z|\geqslant\tau^{-1}$, for every $\tau\in(1, r)$, and so (2.10) follows from (2.17) easily. ∎

3. Zero Distributions

For orthogonal polynomials on the unit circle, Nevai and Totik [14] and Mhaskar and Saff [13] obtained some results about the zero distributions of these polynomials. In their discussion, the recurrence relation played a very important role. In this section, we will prove some results similar to those in [14, 13]. Since there are no recurrence relations available for $\phi_{n,p}(z)$ when $p\neq 2$, we have to use a different method than that in the above cited works.

Theorem 3.1. *Let $p>1$; assume $d\mu$ satisfies the Szegő condition and $[D(d\mu, z)]^{-2/p}$ is not analytic on $\bar{\Delta}:=\{z\in\mathbf{C}:|z|\leqslant 1\}$. Then $\nu(\phi_{n,p})$ converges in the weak-star topology to the uniform distribution on $|z|=1$ for a subsequence $n\in\Lambda\subseteq\mathcal{N}$.*

Here $\nu(\phi_{n,p})$ is the discrete unit measure defined on the Borel set in $\mathbb{C}$ having mass $1/n$ at each zero of $\phi_{n,p}$.

Recall that $\operatorname{cap}(\bar{\Delta})=1$ ("cap" means the logarithmic capacity) and the equilibrium measure of $\bar{\Delta}$, $\mu_{\bar{\Delta}}$, is the uniform distribution on Γ, i.e., $\mu_{\bar{\Delta}}=d\theta/2\pi$ on Γ.

Proof of Theorem 3.1. First, since all the zeros of $\phi_{n,p}$ lie in $|z|<1$, so

$$\nu(\phi^*_{n,p})(A)=0 \tag{3.1}$$

for every A (*closed*) $\subset \Delta$. Since $\phi_{n,p}(0)$ is (plus or minus) the product of the zeros of $\phi_{n,p}$, we have $|\phi_{n,p}(0)|\leqslant 1$, and so $\limsup_{n\to\infty}|\phi_{n,p}(0)|^{1/n}\leqslant 1$. We claim that

$$\limsup_{n\to\infty}|\phi_{n,p}(0)|^{1/n}=1. \tag{3.2}$$

Let us assume there is $R\in(1,\infty)$ such that

$$\limsup_{n\to\infty}|\phi_{n,p}(0)|^{1/n}=1/R.$$

By the definition of $\varepsilon_{n,p}$ and the fact that $z=e^{i\theta}=1/\bar{z}$, we have

$$\begin{aligned}\varepsilon^p_{n,p}&=\inf_{P_{n-1}\in\mathscr{P}_{n-1}}\frac{1}{2\pi}\int_0^{2\pi}|z^n+P_{n-1}(z)|^p\,d\mu(\theta)\\&=\inf_{P_{n-1}\in\mathscr{P}_{n-1}}\frac{1}{2\pi}\int_0^{2\pi}|1+zP_{n-1}(z)|^p\,d\mu(\theta).\end{aligned}$$

Note that

$$\phi^*_{n,p}(z)-\overline{\phi_{n,p}(0)}\,z^n-1\in z\mathscr{P}_{n-2},$$

so we can have

$$\begin{aligned}\varepsilon_{n-1,p}&\leqslant\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi^*_{n,p}(z)-\overline{\phi_{n,p}(0)}\,z^n|^p\,d\mu(\theta)\right)^{1/p}\\&\leqslant\varepsilon_{n,p}+|\phi_{n,p}(0)|\left(\frac{1}{2\pi}\int_0^{2\pi}d\mu(\theta)\right)^{1/p},\end{aligned}$$

therefore

$$\varepsilon_{n-1,p}-\varepsilon_{n,p}\leqslant|\phi_{n,p}(0)|\left(\frac{1}{2\pi}\int_0^{2\pi}d\mu(\theta)\right)^{1/p}.$$

Hence

$$\limsup_{n\to\infty} (\varepsilon_{n-1,p}-\varepsilon_{n,p})^{1/n} \leqslant 1/R,$$

so it follows that

$$\limsup_{n\to\infty} (\varepsilon_{n,p}-[G(d\mu)]^{1/p})^{1/n} \leqslant 1/R. \tag{3.3}$$

With the inequality in the remark after the proof of Theorem 2.2, (3.3) implies that

$$\limsup_{n\to\infty} (\max_{|z|\leqslant\rho} |\Phi^*_{n,p}(z)-\Phi^*_{n+1,p}(z)|)^{1/n} \leqslant 1/R^{1/\tilde{q}},$$

for $\rho\in(0,1)$. Using Bernstein's inequality (cf. [17, p. 77]), we can show that $\Phi^*_{n,p}(z)$ converges locally uniformly for $|z|<R^{1/\tilde{q}}$, and consequently $[D(d\mu, z)]^{-2/p}$ has analytic continuation to $|z|<R^{1/\tilde{q}}$, which contradicts the assumption that $[D(d\mu, z)]^{-2/p}$ is not analytic on $\bar{\Delta}$. This proves our claim (3.2).

Now, by the Bernstein inequality, for every $\rho\in(0,1)$,

$$\max_{|z|\leqslant 1} |\Phi^*_{n,p}(z)| \leqslant \left(\frac{1}{\rho}\right)^n \max_{|z|\leqslant\rho} |\Phi^*_{n,p}(z)|,$$

so

$$\limsup_{n\to\infty} \max_{|z|\leqslant 1} |\Phi^*_{n,p}(z)|^{1/n} \leqslant \frac{1}{\rho},$$

because by Theorem 2.2

$$\lim_{n\to\infty} \Phi^*_{n,p}(z)=[D(d\mu, z)]^{-2/p}$$

locally uniformly for $|z|<1$. Hence, by the arbitrariness of $\rho\in(0,1)$, and together with (2.1) and (3.2), it follows that, for some $\Lambda\subseteq\mathcal{N}$,

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} \left(\max_{|z|\leqslant 1}\left|\frac{\Phi^*_{n,p}(z)}{\phi_{n,p}(0)/\varepsilon_{n,p}}\right|\right)^{1/n} \leqslant 1=\operatorname{cap}(\bar{\Delta}). \tag{3.4}$$

Using Theorem 2.1 in [1] for the monic polynomials

$$\frac{\Phi^*_{n,p}(z)}{\phi_{n,p}(0)/\varepsilon_{n,p}},$$

from (3.1) and (3.4) we get

$$\nu(\Phi_{n,p}^{*}) \to \mu_{\bar{\Delta}}, \qquad n \to \infty, \qquad n \in \Lambda,$$

or equivalently,

$$\nu(\phi_{n,p}) \to \mu_{\bar{\Delta}}, \qquad n \to \infty, \qquad n \in \Lambda. \quad \blacksquare$$

Next, we consider the case when the zeros of $\phi_{n,p}(z)$ stay away from the unit circle as in [14]. Let

$$z_{k,n}^{(p)} := z_{k,n}^{(p)}(d\mu)$$

denote the zeros of $\phi_{n,p}(z)$ ordered in such a way that

$$|z_{n,n}^{(p)}| \leqslant z_{n-1,n}^{(p)}| \cdots \leqslant |z_{1,n}^{(p)}| < 1.$$

THEOREM 3.2. *Let μ satisfy the Szegő condition, and $p > 1$. Then the following assertions are equivalent*:

(a) $\lim \operatorname{dup}_{n \to \infty} |z_{1,n}^{(p)}(d\mu)| < 1$;

(b) $[D(d\mu, z)]^{-2/p}$ *is analytic in* $|z| < r$ *for some* $r > 1$;

(c) $\lim \sup_{n \to \infty} |\phi_{n,p}(0)|^{1/n} < 1$;

(d) $\sup_n \max_{|z| \leqslant \rho} |\phi_{n,p}^{*}(z)| < \infty$ *for some* $\rho > 1$.

Proof. (a) $\Rightarrow$ (b): If $D^{-1}(d\mu, z)$ is not analytic in $|z| < r$ for any $r > 1$, then from Theorem 3.1, we have

$$\nu(\phi_{n,p}) \to \mu_{\bar{\Delta}}, \qquad n \to \infty, \qquad n \in \Lambda \qquad \text{for some } \Lambda \subset \mathcal{N},$$

which contradicts (a).

(b) $\Rightarrow$ (a): Assume $[D(d\mu, z)]^{-2/p}$ is analytic in $|z| < r$ for some $r > 1$. From Theorem 2.4(ii), we have

$$\phi_{n,p}(z) \cong [G(d\mu)]^{1/p} z^n [\bar{D}(d\mu, z^{-1})]^{-2/p}$$

which holds locally uniformly for $|z| > r_2^{-1} > r^{-1}$, so

$$\limsup_{n \to \infty} |z_{1,n}^{(p)}(d\mu)| \leqslant r_2^{-1} < 1.$$

(a) $\Rightarrow$ (c): Note that

$$\phi_{n,p}(0) = \prod_{i=1}^{n} |z_{i,n}^{(p)}| \leqslant |z_{1,n}^{(p)}|^n,$$

so

$$\limsup_{n \to \infty} |\phi_{n,p}(0)|^{1/n} \leqslant \limsup_{n \to \infty} |z_{1,n}^{(p)}(d\mu)| < 1.$$

(c) $\Rightarrow$ (b): The proof is contained in the proof of (3.2).

(d) $\Rightarrow$ (c): Since $\phi_{n,p}(0)$ is the leading coefficient of $\phi^*_{n,p}(z)$, it is easy to see (by maximum principle) that

$$|\phi_{n,p}(0)| \leqslant \frac{1}{\rho^n} \max_{|z| \leqslant \rho} |\phi^*_{n,p}(z)|.$$

So

$$\limsup_{n \to \infty} |\phi_{n,p}(0)|^{1/n} \leqslant \frac{1}{\rho} < 1.$$

(b) $\Rightarrow$ (d): Let $[D(d\mu, z)]^{-2/p}$ be analytic in $|z| < r$ for some $r > 1$, from Theorem 2.4(ii), we have, for some $r_2 \in (1, r)$,

$$\phi_{n,p}(z) = [G(d\mu)]^{1/p} z^n [\bar{D}(d\mu, z)]^{-2/p} + O\left(\frac{1}{r_2^n}\right)$$

locally uniformly for $|z| > r^{-1}$. So

$$\phi^*_{n,p}(z) \to [G(d\mu)]^{1/p} D^{-2/p}(d\mu, z)$$

locally uniformly for $|z| < r_2$, hence $\sup_n \max_{|z| \leqslant \rho} |\phi^*_{n,p}(z)|$ is finite for some $\rho \in (1, r_2)$. ∎

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